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STEADY-STATE SHEAR-BANDS IN THERMO-PLASTICITY—I. VANISHING YIELD STRESS

JOHN H. MADDOCKS
Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, U.S.A.

and

REZA MALEK-MADANI
Department of Mathematics, U.S. Naval Academy, Annapolis, MD 21402, U.S.A.

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Abstract—This paper concerns the construction and stability properties of steady-state solutions of a system of partial differential equations that model simple shearing of a slab of thermo-plastic material. The class of constitutive laws that give rise to a variational formulation of the steady-state problem is identified, and a phase-plane argument is used to construct time-independent solutions that may be interpreted as steady-state shear-bands. Our variational framework captures several commonly adopted constitutive laws. Techniques from bifurcation theory for variational problems are applied to classify stable and unstable solutions merely in terms of the shape of the solution branch in the distinguished bifurcation diagram that arises when average strain-rate is plotted against shearing force. There are two novel features to our approach. First, the two problems in which loading is imposed by either stress boundary conditions or velocity boundary conditions are treated by one analysis, and the differing stability properties of solutions are explained naturally. Second, the stability analysis is based upon a symmetric eigenvalue problem arising from the appropriate second variation. The link with dynamic behavior is made through a Lyapunov functional, and the linearized dynamics are not considered directly. Provided the proper existence theorems for the time-dependent problem can be proven or are assumed, our Lyapunov approach yields the appropriate nonlinear dynamic stability properties of steady-state solutions. In this paper we shall consider the case in which vanishing strain-rate implies zero stress, i.e. there is no residual or yield stress present in our model, but our analysis can be extended to encompass constitutive laws modelling nonzero yield stress.

1. INTRODUCTION

In this paper we shall study an elementary one-dimensional thermo-plastic model of the simple shearing of a solid in motion at a high strain-rate. More particularly, we shall study questions of existence and stability of nonuniform, steady-state solutions of the system of partial differential equations

$$\begin{aligned} \rho v_t &= \sigma_x, \\ \rho c\theta_t &= \lambda\theta_{xx} + \sigma v_x. \end{aligned} \tag{1}$$

Here the unknowns are the velocity $v(x,t)$, the temperature $\theta(x,t)$ and the stress $\sigma(x,t)$, which are each functions of time $t > 0$, and one space dimension $-h < x < h$. The problem parameters are the density ρ, specific heat c, coefficient of thermal conductivity λ and half-width h. (We have set the Taylor–Quinney coefficient to unity.) The material parameters are assumed to be constants, independent of time, space, temperature and velocity. Equations (1) will be interpreted as describing the temperature θ and horizontal velocity v of a long, prismatic slab of material. The cross-section of the slab is implicitly assumed to be a rectangle of dimensions $2h \times d$. The top and bottom edges of the slab are loaded with a constant shear that acts parallel to the axis of the prism. It is assumed that the only spatial dependence of the velocity and temperature arises across the depth of the slab as measured by the variable x, $-h < x < h$. To be compatible with the assumption of no spatial dependence of the solution across the width of the slab, the vertical sides of the slab are assumed to be insulated and unloaded. Equations (1) are completed once a constitutive law

$$\sigma = \sigma(\theta, v_x), \tag{2}$$

and boundary conditions are prescribed. The boundary conditions that will be explicitly considered are the stress boundary conditions

$$\sigma = \sigma_0|_{x=h}, \quad v(-h,t) = 0, \quad \theta = \theta_0|_{x=-h,h}, \quad t > 0, \tag{3}$$

or the velocity boundary conditions

$$v(h,t) = v_0, \quad v(-h,t) = 0, \quad \theta = \theta_0|_{x=-h,h}, \quad t > 0. \tag{4}$$

In both cases the bottom edge of the slab is located in space by the prescription $v(-h,t) = 0$. A shear motion is then induced by the imposition of either a stress or a velocity at the top edge.

It is convenient to nondimensionalize the problem with h, $c\rho h^2/\lambda$, $\rho d h^2$ and θ_0 as units of length, time, mass and temperature, respectively. Then eqns (1)–(4) can be rewritten in the nondimensional form:

$$v_t = \sigma_x,$$
$$\theta_t = \theta_{xx} + \kappa\sigma v_x, \tag{5}$$

$$\sigma = \sigma(\theta, v_x), \tag{6}$$

$$\sigma = \sigma_0|_{x=1}, \quad v(-1,t) = 0, \quad \theta = 0|_{x=-1,1}, \quad t > 0, \tag{7}$$

$$v(1,t) = v_0, \quad v(-1,t) = 0, \quad \theta = 0|_{x=-1,1}, \quad t > 0. \tag{8}$$

In terms of dimensional quantities the nondimensional parameter κ is

$$\kappa \equiv \frac{\lambda^2}{c^3\rho^2 h^2 \theta_0}. \tag{9}$$

As we shall henceforth not consider dimensional quantities, the same symbols have been retained for the associated nondimensional variables.

System (5) arises as a model in several areas of continuum mechanics. Of course the underlying application determines the appropriate arguments and qualitative form of the constitutive relation for the stress σ. In fluid dynamics, eqns (5) with constitutive relation $\sigma = k e^{-\alpha\theta} v_x$ have been used as a simple model for Couette flow. The question of stability of the steady-state solutions in this context has been addressed by many authors, including Regirer (1957), Joseph (1964) and Mazo and Ruderman (1986) among others. Equations with a superficial similarity to system (5) also occur in the theory of combusion [cf. Zeldovich *et al.* (1985)]. When attention is restricted to the so-called "intermediate-asymptotic theory" (Zeldovich *et al.*, 1985), the system of equations describing combustion reduce to a form that is mathematically equivalent to the steady-state version of the system (5). The phenomenon of multiplicity of solutions was observed in this combustion theory context for a particular exponential nonlinearity, and stability questions have also been addressed (Gel'fand, 1963, Zeldovich *et al.*, 1985). Our work differs significantly because in order to model thermo-plasticity more general nonlinearities must be allowed in constitutive relation (6). As a consequence the steady-state equations are more complicated. Furthermore, the first of the dynamic equations (5) depends quasi-linearly on velocity, whereas the corresponding equation in combustion theory has a simpler semi-linear form.

In this presentation the application we have in mind is the modelling of the plastic behavior of solids. The literature concerning such high strain-rate deformations is extensive. The assumption of high strain-rate enters in part because elastic effects, i.e. dependence of the constitutive relation upon strain, are neglected, and attention is focussed upon plastic

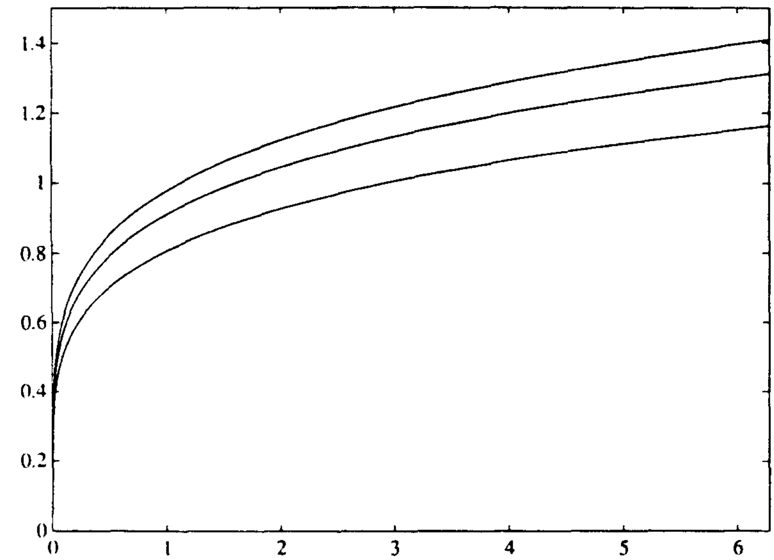

Fig. 1. Typical stress (ordinate) versus strain-rate (abscissa) curves for three fixed values of the nondimensional temperature θ. The particular plots shown are for the constitutive function $g(p)$ defined by

$$g(x) = \nu x^p,$$

where the choice of constants $\nu = 1$, $\alpha = 0.02$ [cf. (15)] and $p = 0.2$ are motivated by the experimental data furnished in Wright (1987).

behavior due to dependence of the stress upon the strain-rate v_x. The model assumed here, or simple variants of it, have been investigated recently by Burns and Trucano (1982), Burns (1985), Wright and Batra (1985), Wright (1987), Fressengas and Molinari (1987), Anand *et al.* (1987), Tzavaras (1986, 1987), Chen *et al.* (1989) and Molinari and Leroy (1990). The main point of interest in these works is an understanding of the mechanisms that lead to *shear-band* solutions. In shear-bands the velocity gradient is extremely high within a narrow band of the material, and small outside it. Such configurations have been observed experimentally [cf. Rogers (1979), Hutchinson (1984), Marchand and Duffy (1988) and Needleman and Tvergaard (1987)]. One interpretation of the mechanism for the formation of shear-bands is that the plastic work generates heat which thermally softens the material. This softening is thought to overcome any strain-hardening or strain-rate stiffening effects. In our notation strain-hardening would be modelled by dependence of constitutive relation (6) on the deformation gradient. Here we neglect strain-hardening in deference to strain-rate effects, which are modelled by the dependence of constitutive relation (6) upon the velocity gradient v_x. In the presence of strain-hardening effects steady-state solutions are not possible, so that, as is commonplace (cf. the references cited above), our steady-state model should properly be interpreted as describing intermediate time behavior that arises before effects such as fracture dominate.

The qualitative form of the stress vs strain-rate constitutive relations that are assumed here are depicted in Fig. 1. In common with all the above authors we assume that vanishing strain-rate implies zero stress. Thus the model at low strain-rates is of questionable validity. In Part II of this paper (in preparation) we extend our techniques to constitutive relations that retain a vestige of elastic effects through a non-zero yield stress, which we interpret as the limiting positive value of stress as the strain-rate is decreased to zero (with temperature held fixed). However the analysis and results in this extended model are considerably more intricate, and in Part I we have chosen to first introduce our solution techniques and results in the context of the more standard and simple model.

In the case of adiabatic problems—in which the boundary conditions on θ appearing in (7)-(8) are changed to model thermally insulated walls, or the extreme case of a nonconducting material with $\lambda = 0$—a typical analysis has been limited to showing that the steady, uniform-shear solution is unstable at high strain-rates. Perturbation analysis has been the main analytical tool used in such models of the prediction of the onset of shear localization. For example, Bai (1982) and Shawki *et al.* (1983) examined an infinite domain undergoing simple shearing, while Douglas and Chen (1985) studied adiabatic anti-plane shear. Burns and Trucano (1982) considered a plate of rather general material under simple shear with insulated (adiabatic) thermal boundary conditions, and constant velocity imposed at the upper edge of the plate. In the case of adiabatic boundary conditions, truly steady-state solutions cannot arise due to the generation of heat which cannot escape through the boundary. Accordingly, the perturbation analyses of Burns and Trucano (1982) and Burns (1985) are based upon the assumption that the basic solution varies slowly in time. The results they obtain agree well with the experimental observations of localization described by Costin *et al.* (1979). Subsequently, Fressengas and Molinari (1987) introduced another perturbation method to account for a more pronounced time-variation of the basic solution. In a recent paper Molinari and Leroy (1990) investigate the steady-state solutions of the governing equations (1)–(2) when the stress–strain law is a monomial in $e^{-\alpha\theta}v_x$, and the boundary conditions are mixed in velocity and stress. Consequently the class of constitutive relations assumed by Molinari and Leroy is considerably more restrictive than that treated here, although they do allow more general boundary conditions. Their approach also involves the further approximation of a quasi-static linear perturbation analysis (i.e. they consider only the linearized system in which the acceleration term ρv_t is neglected) to classify the stable and unstable steady-state solutions in a rather formal sense. In the cases where the two models overlap, the predictions of stability properties found by the separation of variables analysis of Molinari and Leroy coincide with the more rigorous predictions arising in the variational analysis that we follow.

In the case of a nonconducting material, Tzavaras (1986) studied the adiabatic shearing of a non-Newtonian material with temperature-dependent viscosity, and proved that a uniform shearing solution of system (1) with $\lambda = 0$ is asymptotically stable as $t \to \infty$. In a more striking result, Tzavaras (1987) considered the adiabatic plastic shearing of an infinite slab subjected to prescribed traction at the boundaries. By assuming that thermal softening prevails over strain hardening, he classified a set of initial conditions for which a classical solution ceases to exist within a *finite* time.

In contrast to the adiabatic models assumed in the above works, our thermal boundary conditions approximate an enveloping heat bath. Consequently truly steady-state solutions of (5)–(8) can and do arise. It should be remarked that it is a point of some debate as to whether or not steady-state shear bands have time to arise in actual experiments before other phenomena, such as fracture, dominate the physical behavior. In either case we believe an understanding of steady-state solutions to be important because the critical points have a large bearing upon the dynamics of the time-dependent parabolic system (5), which is generally accepted as a reasonable model on some time scale.

While difficulties associated with time-dependence of the basic solution do not arise in our model, the solutions that we examine have nontrivial *spatial* dependence. Consequently, a stability analysis based upon linearization and separation of the time variable would involve a nonsymmetric eigenvalue problem for a system of differential equations with nonconstant coefficients. A direct assault on such problems can be quite complicated [cf. Chen *et al.* (1989) and Molinari and Leroy (1990)]. In the present paper we employ an intrinsically different strategy that is based upon ideas arising from bifurcation theory for problems with an underlying variational structure. The new techniques apply to only a restricted class of constitutive relations, but strengthened results are obtained for this somewhat smaller class of materials. Moreover, the simplest, standard constitutive relations fall within the purview of our variational analysis. First a phase-plane argument is used to construct all steady-state solutions of (5)–(8) in cases that arise when the form of constitutive relation (6) is such that the steady-state problem possesses an underlying variational structure. We then apply results of Maddocks (1987), which provide a characterization of those extremals that are actually local minima purely in terms of the shape of the branch of solutions in a certain bifurcation diagram.

The cases of both stress boundary conditions (7), and velocity boundary conditions (8) will be treated. As in the nonvariational problem, there is a multiplicity of equilibrium (steady-state) solutions of (5)–(8), dependent upon the range of the nondimensional parameters σ_0, v_0 and κ. In the variational formulation, the steady-state solutions are also extremals of a certain potential. For boundary conditions (7) an unconstrained variational principle is appropriate, but boundary conditions (8) are encompassed by consideration of a conditional variational principle involving an isoperimetric side constraint. In either case, the results of Maddocks (1987) can be exploited. Moreover, the potential, or Lagrangian, is a Lyapunov functional whose time derivative along solutions of (5)–(8) is nonpositive. Concomitantly, provided that an appropriate existence result for the time-dependent system is either known or is assumed, we obtain nonlinear, dynamic stability of steady-state solutions that are local minima. Because the dynamics are dissipative, instability of steady-state solutions that are other types of extremals can also be expected. While it is apparent, on physical grounds, that the set of all solutions is the same for both boundary-value problems, our variational analysis clarifies the connections between the two loading mechanisms and also contrasts the differences in stability properties that arise in the two problems.

Our conclusions are as follows. For the class of constitutive relations detailed in eqns (15) and (26) (see also Fig. 1), there is a one-parameter family of steady-state solutions of (5) and (6), subject to either boundary conditions (7) or (8). The maximum temperature of the solution, θ_m say, provides a smooth parametrization of a solution branch along which the boundary data σ_0 or v_0 must be chosen appropriately. There are no other steady-state solutions. The steady-state velocity at the top boundary is an increasing function of the branch parameter θ_m, and consequently boundary conditions (8) with a prescribed value of v_0 yield a unique solution. The boundary stress does not depend on θ_m monotonically. In fact for σ_0 sufficiently large, (5)–(7) has no solution, for σ_0 sufficiently small there are two solutions, and for intermediate values there may be any even number of solutions. The exact number depends upon the precise constitutive relation that is assumed. A representative bifurcation diagram is depicted in Fig. 4b, where v_0 has been plotted against σ_0. Our stability results can be summarized in terms of the solution branches in this figure. It will be convenient to describe segments of a solution branch on which σ_0 increases with increasing v_0 as *forward-going*, and segments on which σ_0 decreases with increasing v_0 as *backward-going*. Our analysis will demonstrate that the entire branch is stable whenever the velocity boundary conditions (8) are enforced. In contrast only the forward-going segments are stable when the loading mechanism is such that the stress boundary conditions (7) are appropriate. The precise sense in which we use the adjective stable will be further described in Section 5.

The remainder of the presentation has the following structure. Section 2 classifies those stress–strain laws $\sigma(\theta, v_x)$ that admit a variational formulation of the steady-state problem. A summary of the results of Maddocks (1987) that will be employed here is given in Section 3. The core of our analysis is contained in Section 4, which describes the detailed phase-plane arguments that provide the bifurcation diagrams (Figs 3 and 4b) for the steady-state problem. In Section 5, the appropriate Lyapunov stability results are outlined. Our results are summarized in Section 6.

2. THE VARIATIONAL FORMULATION

In this section we shall determine a class of constitutive relations (6) that is compatible with a variational formulation of the steady-state problem. The analysis is a simple example of what is sometimes called the inverse-problem of the calculus of variations, namely find a functional that gives rise to prescribed Euler–Lagrange equations. It will be shown that a variational principle arises only if the dependencies of the constitutive law upon the temperature and strain-rate are coupled in a very particular way. We seek a function $\Sigma = \Sigma(\theta, v_x)$, and a (positive) constant α with the property that the Euler–Lagrange equations of the functional

$$I(\theta, v_x, \sigma_0) = \int_{-1}^{1} \left[\frac{1}{2}\frac{\alpha}{\kappa}\theta_x^2 + \Sigma(\theta, v_x) - \sigma_0 v_x\right] \mathrm{d}x,$$

$$v(-1) = 0, \quad \theta(-1) = \theta(1) = 0. \tag{10}$$

coincide with the time-independent version of eqns (5). Because θ_x occurs in neither (5) nor (6), the integrand of (10) can have no more complicated dependence on θ_x. The dependence of (10) on θ and v_x is for the moment arbitrary, but the particular form written here will simplify subsequent calculations. The third term in the integrand of (10) accounts for the fourth boundary condition. If σ_0 is specified, then the natural boundary condition at $x = 1$ will complete the stress boundary conditions (7) (*vide infra*). The velocity boundary conditions (8) are recovered if σ_0 is interpreted as a Lagrange multiplier associated with the constraint $v(1) - v(-1) = v_0$.

The Euler–Lagrange equations of (10) are

$$-\frac{\alpha}{\kappa}\theta_{xx} + \Sigma_\theta = 0,$$

$$(\Sigma_{v_x} - \sigma_0)_x = 0. \tag{11}$$

Comparison with the steady-state equations arising from (5) and the boundary conditions, then provides the relations

$$\Sigma_\theta = -\alpha\sigma v_x,$$

$$\Sigma_{v_x} = \sigma. \tag{12}$$

When σ is eliminated from eqns (12), a first-order partial differential equation for Σ is obtained:

$$\Sigma_\theta + \alpha v_x \Sigma_{v_x} = 0. \tag{13}$$

The general solution of (13) can easily be found by the method of characteristics [see, e.g. Courant and Hilbert (1962)]. The solution is

$$\Sigma(\theta, v_x) = G(\mathrm{e}^{-\alpha\theta} v_x), \tag{14}$$

where G is an arbitrary smooth function of a single variable. Consequently (14) gives the most general form of the function Σ for which (10) provides a variational formulation of the steady-state problem. From (12) it follows that the associated constitutive relation is

$$\sigma(\theta, v_x) \equiv \mathrm{e}^{-\alpha\theta} g(\mathrm{e}^{-\alpha\theta} v_x), \tag{15}$$

where $g(p) \equiv G'(p)$. A detailed discussion of the appropriate assumptions on $g(p)$ will be provided later, but we remark in passing that the Clausius-Duhem inequality is satisfied by the constitutive law (15) provided that $\alpha > 0$ and $g(p)$ is a monotone increasing function.

The choice $g(p) \equiv p$ was used in the papers of Joseph (1964) and of Mazo and Ruderman (1986) in connection with Couette flow, and by Chen *et al.* (1989) in modelling shear-band type, steady-state solutions in solids. Molinari and Leroy (1990) assume that $g(p) = p^a$, i.e. a simple monomial. Accordingly, the stability results that will be obtained in this paper include, and generalize, the results of the above-mentioned papers that concern the same boundary conditions.

3. A VARIATIONAL STABILITY THEOREM

This section summarizes the theory developed in Maddocks (1987) that will be exploited in the stability arguments of Section 5. The most general case will not be described; rather the analysis will be specialized to a framework appropriate to the example at hand, namely the steady-state solutions of (5). The central idea is this: in specific circumstances the properties of the second variation at solutions of a one-parameter family of variational principles can be analysed exhaustively using only information contained in a certain distinguished bifurcation diagram.

Consider a one-parameter family of variational principles of the standard calculus of variations type

$$\min_{y \in B} I(y, \gamma) \equiv \int_a^b F(x, y, y', \gamma)\,\mathrm{d}x. \tag{16}$$

Here $y(x)$ is a vector function of the scalar variable x, y' is the vector of derivatives with respect to x, γ is a scalar parameter, B denotes an appropriate set of functions (including boundary conditions) and $F(x, y, p, \gamma)$ is a smooth function from $\Re \times \Re^n \times \Re^n \times \Re$ to $\Re$. The derivation of the first- and second-order conditions associated with (16) is completely standard. See, for example, Gel'fand and Fomin (1963) or Cesari (1983). In particular, for given γ, the first-order conditions include the system of Euler–Lagrange equations

$$-(F_p(x, y, y', \gamma))_x + F_y(x, y, y', \gamma) = 0, \tag{17}$$

and possibly, dependent upon the precise form of the set B, some natural boundary conditions. We shall assume that eqns (17) have been solved for a family of solutions, or *extremals*, $(\hat{y}(s), \hat{\gamma}(s))$ with $\hat{y} \in B$. Here s denotes an arbitrary, smooth, nonsingular parameterization of the branch of extremals, i.e. $\dot{\hat{y}}^2 + \dot{\hat{\gamma}}^2 \neq 0$ for any value of s.

The second-order conditions guaranteeing that an extremal is actually a local minimum are of two types. First, it will be assumed that Legendre's strengthened condition holds everywhere, that is the matrix of second partials with respect to the derivative variable is everywhere positive definite,

$$F_{pp} > 0. \tag{18}$$

The remaining second-order condition reduces to a study of the symmetric eigenvalue problem associated with Jacobi's accessory equation, namely

$$-(\hat{F}_{pp} u_x + \hat{F}_{py} u)_x + \hat{F}_{py}^T u_x + \hat{F}_{yy} u = \mu u, \tag{19}$$

subject to the appropriate homogeneous, linearized boundary conditions. Here $u(x)$ is the eigenfunction, μ is the eigenvalue, and the coefficients $\hat{F}_{pp}$, $\hat{F}_{yy}$ and $\hat{F}_{py}$ denote the appropriate second-order derivative matrices evaluated on the extremal $(\hat{y}, \hat{\gamma})$. If (19) has only positive eigenvalues, the associated second variation is a positive-definite quadratic form, and consequently the extremal $(\hat{y}, \hat{\gamma})$ is actually a local minimum (in the $C[a, b]$ topology).

The special structure that will be exploited here is that a complete *family* of extremals is under consideration, and the Euler–Lagrange equations (17) can be regarded as defining a bifurcation problem with an associated variational structure. Such problems have two special features. First, Hopf bifurcations cannot arise, and second, as a branch of extremals is traversed, the eigenvalues of (19) can only change sign when there is a bifurcation point (or singularity) in the family of extremals [Krasnoselskii's Theorem, see e.g. Zeidler (1985), Kielhofer (1988)]. In the absence of symmetry the only structurally stable, or robust, singularities are (simple) *fold points*, and we shall assume that no other bifurcations arise. A fold point is characterized by the property that it is an extremal for which the parameter $\hat{\gamma}(s)$ realizes either a local minimum or maximum along the branch of extremals. Consequently a fold point separates the branch of extremals into two segments both of which lie (locally) either to the left or right of the fold point in any bifurcation diagram with γ plotted as abscissa (cf. the solution branches in Figs 3 and 4b). The two cases will be referred to as subcritical and supercritical folds, respectively.

The principle of exchange of stability is a classic result tracing back at least to Poincaré (1885). It implies the following: whenever one branch of a simple fold is stable, that is whenever (19) has only positive eigenvalues, then on the other segment (19) has at least one negative eigenvalue. The results derived in Maddocks (1987) are related, but provide strengthened conclusions that are valid only when the bifurcation problem can be associated with a variational principle. In particular, stability of one segment of the fold does not have to be taken as a hypothesis. Instead it is observed that there is a distinguished functional, namely

$$-I_{\gamma}(y,\gamma) \equiv -\int_a^b F_{\gamma}(x,y,y',\gamma)\,\mathrm{d}x, \tag{20}$$

with special properties. Explicitly, *in a bifurcation diagram with* $-I_{\gamma}(\hat{y},\hat{\gamma})$ *plotted as ordinate and* γ *as abscissa*, eigenvalue problem (19) on an *upper branch*, adjacent to a *subcritical* fold has at least one negative eigenvalue. Similarly on the *lower* branch adjacent to a *supercritical* fold eigenvalue problem (19) has a negative eigenvalue. These strengthened variational results are particularly useful on solution branches that are folded several times.

The idea underlying the proof is simple. Differentiation of (17) with respect to s reveals that when $\dot{\gamma} = 0$—e.g. at a fold point—zero is an eigenvalue of (19). Moreover it can be shown that the sign of the derivative of the critical eigenvalue with respect to the parameterization s is encoded in the shape of the solution branch in the $(\gamma, -I_{\gamma})$ bifurcation diagram. It happens that as a subcritical fold is traversed upward, an eigenvalue of (19) crosses from the positive half-line through zero to the negative half-line, and *vice versa* for a supercritical fold.

If degenerate cases involving eigenvalues that only touch zero can be excluded, and the possibility of zero eigenvalues of (19) with multiplicity greater than one can also be ruled out, then more information can be extracted. In such felicitous circumstances precisely one eigenvalue passes through zero at each fold point, and the direction of crossing is completely determined from the distinguished bifurcation diagram. For our purposes it suffices to remark that the degenerate case of an eigenvalue touching zero and not passing through is impossible provided that the branch of extremals in the $(\gamma, -I_{\gamma})$ diagram is smooth with no cusps. The phase-plane analysis described in Section 4 implies the desired properties for the examples of this paper. The same phase-plane analysis also demonstrates that the only bifurcations are fold points. However, the possibility of multiple eigenvalues must be precluded by direct calculation. The required analysis will be described in Section 5.

In many physical problems, including the one considered in this presentation, the parameter γ enters the problem linearly, and the integral appearing in (16) is of the special form

$$I(y,\gamma) \equiv \int_a^b [G(x,y,y') - \gamma H(x,y,y')]\,\mathrm{d}x. \tag{21}$$

Then the distinguished bifurcation diagram is a $(\gamma, \int_a^b H)$ plot. Such diagrams often have physical significance. For example a load vs displacement diagram is obtained for applications in elasticity with uniaxial dead loading. For the plasticity problem (5)–(7) it will later be seen that the distinguished bifurcation diagram is a plot of loading force vs (average) strain-rate.

It is well appreciated that the precise manner in which external loads are imposed has a considerable influence on the stability properties of physical systems. In typical *dead* or *soft* loading an external force is prescribed. The associated variational principles are often of the type (16), with I of the special form (21), and γ plays the role of a specified loading parameter. On the other hand in typical *displacement* or *hard* loading, some generalized displacement is imposed. The associated variational characterizations are then usually *constrained* variational principles. For example

$$\min_{y \in B} \int_a^b G(x, y, y')\,dx, \tag{22}$$

subject to the side condition

$$J(y) \equiv \int_a^b H(x, y, y')\,dx = K, \tag{23}$$

where K denotes some prescribed value of the generalized displacement. The exact type of loading does not affect the set of all equilibria, or extremals. In the constrained variational principle the parameter γ enters as an undetermined Lagrange multiplier associated with constraint (23), and the variational principle again has (21) as Lagrangian. As all values of γ are scanned, all realizable values of K are achieved, and *vice versa*. Accordingly the distinguished (γ, J) bifurcation diagrams for the two problems are identical.

When an extremal is actually a local minimum it can reasonably be anticipated that a stability result will be forthcoming. For a dead-loading problem with an associated variational principle of the type (16), we have already seen that the local minima are determined by eigenvalue problem (19), and, in turn, the eigenvalue problem can be analysed purely from knowledge of the distinguished (γ, J) bifurcation diagram. The stability of steady-state solutions of (5) and (6) subject to boundary conditions (7) will be analysed in this fashion.

For a displacement loading problem with an associated constrained variational principle of the type (22) and (23), the requirements for a local minimum are less stringent [see, for example, Hestenes (1966) or Luenberger (1969)]. Explicitly the second variation of (21) need only be definite for variations $h(x)$ that satisfy the linearized constraint

$$\int_a^b [H_p(x, \hat{y}, \hat{y}')h' + H_y(x, \hat{y}, \hat{y}')h]\,dx = 0. \tag{24}$$

Consequently there can be constrained local minima at which (19) has a negative eigenvalue. In Section 5 of Maddocks (1987) it is shown that the set of all constrained local minima comprises the union of the set of unconstrained local minima, with the set of extremals at which (19) has one negative eigenvalue *and the derivative of J with respect to γ along the solution branch is negative*

$$\frac{\partial J}{\partial \gamma} < 0. \tag{25}$$

But $\partial J/\partial \gamma$ is the slope of the solution branch in the distinguished (γ, J) bifurcation diagram. Consequently the second-order conditions of the constrained variational principle can again be analysed once the (γ, J) bifurcation diagram has been calculated. This is the result that will be used to determine the stability properties of the steady-state solutions of (5) and (6) subject to boundary conditions (8).

We shall make use of one more general observation.

Lemma 1. *If the (unconstrained) second variation is positive definite at an extremal of* (21), *then $\partial J/\partial \gamma$ is positive at that solution.*

In one guise or another this result is widely known. One demonstration is provided in Lemma 5.1 of Maddocks (1987).

4. THE BIFURCATION DIAGRAM

The purpose of this section is to construct the bifurcation diagram for solutions of (5)–(8) when the constitutive relation (6) is of the special variational form (15). With no real loss of generality we shall restrict attention to solutions with $v_x \geqslant 0$, so that the constitutive function need only be defined for non-negative arguments. It will be assumed throughout that the function $G(p)$ is twice continuously differentiable for $p > 0$ (and twice continuously right differentiable at $p = 0$), and that $g(p) \equiv G'(p)$ satisfies the constitutive hypotheses

$$g(0) = 0, \quad g'(p) > 0, \quad \forall p \geqslant 0. \tag{26}$$

In Part II of this paper, the assumption that $g(0) = 0$ will be relaxed to $g(0) = g_0 \geqslant 0$, and g_0 will be interpreted as a model of the limiting plastic yield stress. The steady-state version of (5), or equivalently the Euler–Lagrange equations of functional (10), reduce to

$$\begin{gathered}(e^{-\alpha\theta} g(e^{-\alpha\theta} v_x) - \sigma_0)_x = 0,\\ \theta_{xx} + \kappa e^{-\alpha\theta} g(e^{-\alpha\theta} v_x) v_x = 0.\end{gathered} \tag{27}$$

Then θ_{xx} is negative, so that θ realizes its minimum on the boundary. From the non-dimensional boundary conditions (7) or (8) this minimum is $\theta_{min} = 0$.

The first of eqns (27) can be integrated to yield

$$e^{-\alpha\theta} g(e^{-\alpha\theta} v_x) = \sigma_0, \quad x \in [-1, 1]. \tag{28}$$

At first sight it would seem that an additional constant of integration should appear in (28), but this is not so. In the case of stress boundary conditions (7), σ_0 is prescribed, and the natural boundary conditions associated with (10) imply that the additional constant of integration vanishes. In the case of velocity boundary conditions (8), σ_0 is undetermined and can therefore be regarded as the integration constant.

Our next objective is to invert eqn (28), and thereby eliminate the unknown v_x. By constitutive hypotheses (26), the function g is monotone increasing with domain $[0, \infty)$. It will be further assumed that g has range $[0, \infty)$, i.e. $\lim_{p\to\infty} g(p) = \infty$. Then g has an associated continuously differentiable, monotone inverse-function, denoted f, with domain $[0, \infty)$ and range $[0, \infty)$. Now (28) can be inverted to yield

$$v_x = e^{\alpha\theta} f(\sigma_0 e^{\alpha\theta}). \tag{29}$$

In light of (29), eqns (27) are reduced to

$$\theta_{xx} + \kappa\sigma_0 e^{\alpha\theta} f(\sigma_0 e^{\alpha\theta}) = 0. \tag{30}$$

After multiplication by $(\alpha/\kappa)\theta_x$, (30) can be integrated to yield the first integral

$$\frac{1}{2}\frac{\alpha}{\kappa}\theta_x^2 + F(\sigma_0 e^{\alpha\theta}) = A, \tag{31}$$

where F is an anti-derivative of f, and A is a constant of integration. Because f is monotone,

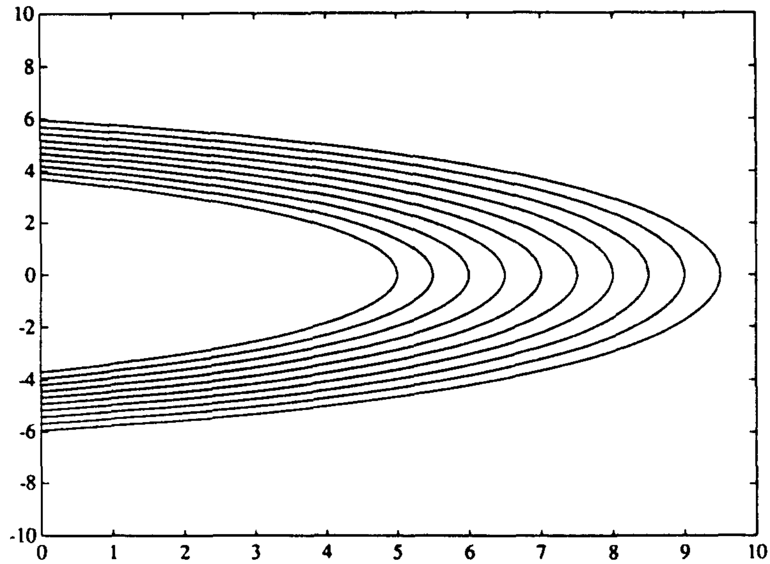

Fig. 2. The phase plane for eqn (30). The graphs are trajectories in the plane with (θ, θ_x) as abscissa and ordinate respectively. The particular function f that we have chosen for this example is the inverse of the constitutive function g described in the caption of Fig. 1, namely

$$f(x) = x^5.$$

The phase plane with $\sigma_0 = 1$ is shown.

the function F is necessarily convex. Accordingly the phase-plane is of the qualitative form depicted in Fig. 2.

The steady-state boundary conditions on θ are

$$\theta(-1) = \theta(1) = 0. \tag{32}$$

Consequently a trajectory in the phase-plane is a solution of the two-point boundary value problem provided that it starts on the line $\theta = 0$ when $x = -1$ and recrosses the same line when $x = 1$. In other words we must select the trajectory that has the correct transit "time". This requirement implies a dependence between A and σ_0. That dependence will now be determined explicitly.

It is apparent from the phase-plane that any solution $\theta(x)$ is even about $x = 0$, and that the maximum value of θ occurs at $x = 0$. Denote this maximum value by θ_m. Then $A = F(\sigma_0\, e^{\alpha\theta_m})$, and (31) can be rewritten

$$\tfrac{1}{2}\alpha\theta_x^2 = \kappa \int_{\sigma_0 e^{\alpha\theta}}^{\sigma_0 e^{\alpha\theta_m}} f(\tau)\, d\tau. \tag{33}$$

By symmetry, a trajectory will satisfy the two-point boundary conditions if

$$1 = \int_{-1}^{0} dx = \int_{0}^{\theta_m} 1 \Big/ \frac{d\theta}{dx}\, d\theta.$$

But (33) allows $d\theta/dx$ to be eliminated, which provides the equation

$$\int_0^{\theta_m} \frac{d\theta}{\left(\int_{\sigma_0 e^{\alpha\theta}}^{\sigma_0 e^{\alpha\theta_m}} f(\tau)\, d\tau\right)^{1/2}} = \sqrt{\frac{2\kappa}{\alpha}}. \tag{34}$$

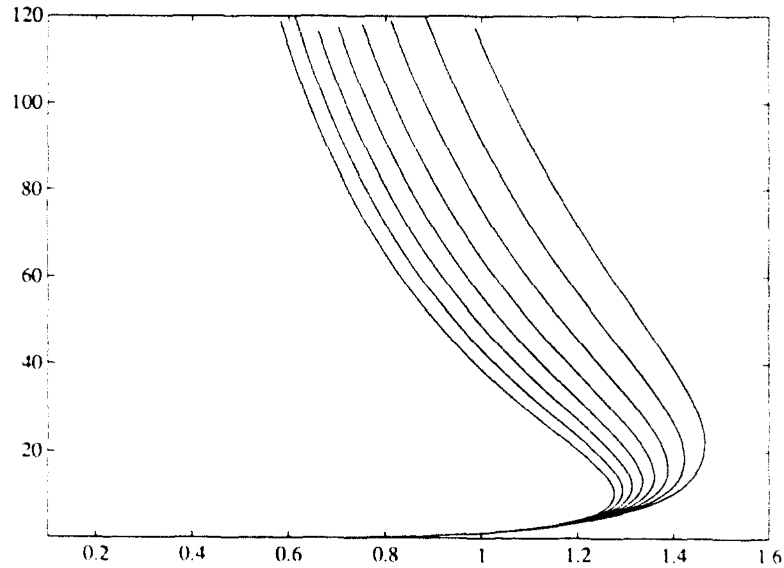

Fig. 3. The bifurcation diagram for the inverse function f described in the caption of Fig. 2. The solution branches are drawn in the preferred coordinate system with σ_0 as abscissa and v_0 as ordinate. Each solution branch is for a particular value of the parameter κ [cf. eqn (10)] in the range [1.0, 2.4].

When the constitutive function is quadratic, i.e. $G(p) \equiv \frac{1}{2}p^2$, then f is a linear function, and eqn (34) can be explicitly integrated to provide the condition

$$\log\left[\frac{e^{\alpha(\theta_m-1)}+\sqrt{e^{2\alpha(\theta_m-1)}-1}}{e^{\alpha(\theta_m-1)}}\right] = \sigma_0\sqrt{\frac{\kappa}{\alpha}}, \tag{35}$$

[cf. the analyses of Joseph (1964), Mazo and Ruderman (1986) and Chen *et al.* (1989)]. Equation (35) defines σ_0 as a function of $\theta_m \in [0, \infty)$. Accordingly θ_m can be used as a parametrization of a unique branch of solutions in a bifurcation diagram with any quantity plotted as ordinate. A bifurcation diagram with the velocity v_0 of the top boundary chosen as ordinate is depicted in Fig. 3. The solution branches are parametrized by the maximum temperature $0 \leqslant \theta_m < \infty$. The pertinent features are that on each branch there are two nonuniform steady-state solutions for σ_0 sufficiently small, no solution for σ_0 large, and $\sigma_0(\theta_m) \to 0$ in either of the limits $\theta_m \to 0$ or $\theta_m \to \infty$. The work of Chen *et al.* (1989) demonstrated *via* direct calculation of the linearized dynamics that the steady-state configuration corresponding to the lower branch is stable when regarded as a solution to the full system of partial differential equations with stress boundary conditions (7). Molinari and Leroy (1990) used a quasi-static approximation to reach similar conclusions (depending upon the precise boundary conditions at hand) for the constitutive law $g(p) = p^q$ (in our notation).

Whenever f is not linear—that is when G is not a quadratic—eqn (34) can no longer be integrated explicitly. Nevertheless, we shall show that several of the qualitative features of the linear case persist. More specifically θ_m can be used to parametrize a unique branch of solutions, $\sigma_0(\theta_m) \to 0$ when $\theta_m \to 0$, and $\sigma_0(\theta_m) \to 0$ as $\theta_m \to \infty$. These claims will be proven in two Lemmata presented below. It should, however, be remarked that in the nonlinear case the solution branch can have multiple folds, rather than the single fold which arises when f is linear. For example Fig. 4b illustrates the solution curves that arise for a given *piecewise* linear constitutive function f for various values of the parameters α and κ.

Our analysis will be facilitated by the introduction of the notation

$$T(\theta_m, \sigma_0) \equiv \int_0^{\theta_m} \frac{d\xi}{\left(\int_{\sigma_0 e^{\alpha\xi}}^{\sigma_0 e^{\alpha\theta_m}} f(\tau)\, d\tau\right)^{1/2}}. \tag{36}$$

The function $T(\sigma_0, \theta_m)$ is (up to a constant factor) the time-map from $\theta = 0$ to $\theta = \theta_m$ on the trajectory with integral constant $A = F(\sigma_0 e^{\alpha\theta_m})$.

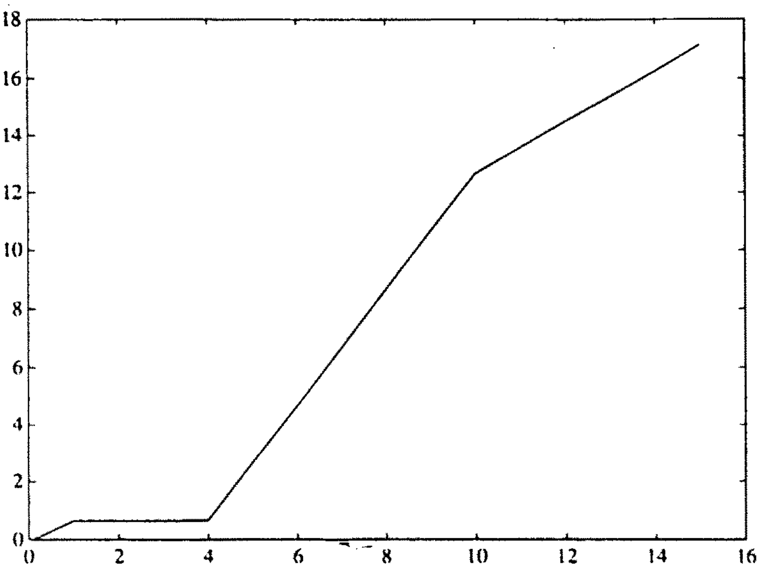

Fig. 4a. An inverse function $f(x)$ defined in terms of piecewise linear functions:

$$f(x) = \begin{cases} 0.7x, & \text{if } 0 \leqslant x \leqslant 1; \\ 0.005(x-1)+0.7, & \text{if } 1 \leqslant x \leqslant 4; \\ 2(x-4)+0.715, & \text{if } 4 \leqslant x \leqslant 10; \\ 0.9(x-10)+12.715, & \text{if } x \geqslant 10. \end{cases}$$

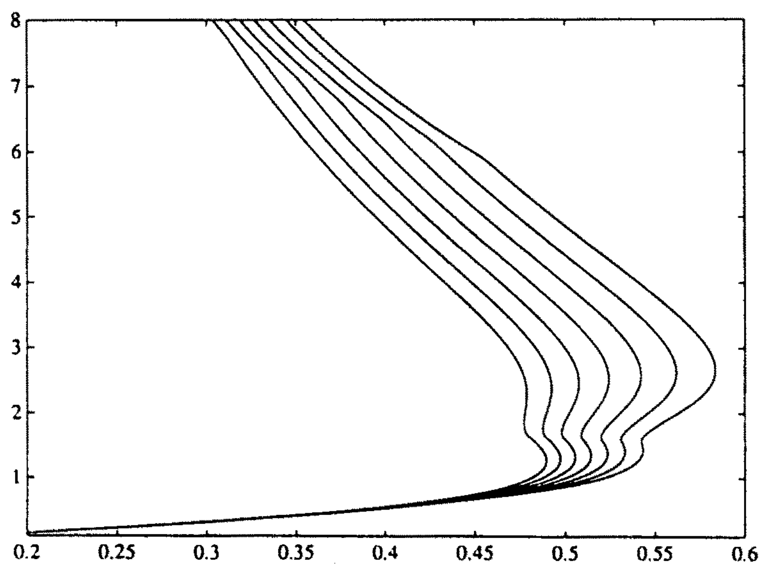

Fig. 4b. The bifurcation diagram analogous to that shown in Fig. 3, but now corresponding to the inverse function f defined in Fig. 4a.

Lemma 2. *For fixed* κ *and* α *(positive), eqn* (33) *uniquely defines a positive function* $\sigma_0(\theta_m)$, $0 < \theta_m < \infty$.

Proof. Equation (34) is a relation of the form

$$T(\theta_m, \sigma_0) = \text{const.} \tag{37}$$

We shall show that

$$\frac{\partial T}{\partial \sigma_0} < 0, \quad \lim_{\sigma_0 \searrow 0} T(\theta_m, \sigma_0) = \infty, \quad \text{and} \quad \lim_{\sigma_0 \to \infty} T(\theta_m, \sigma_0) = 0. \tag{38}$$

The properties described in (38) are sufficient to allow a global implicit function theorem to be applied to (37). It may thereby be concluded that to each given value of $\theta_m > 0$, there is a unique positive value of $\sigma_0 \in (0, \infty)$ such that eqn (37) holds. In other words the conclusions of the Lemma follow.

Direct calculation provides the expression

$$\frac{\partial T}{\partial \sigma_0} = -\frac{1}{2}\int_0^{\theta_m} \frac{f(\sigma_0 e^{\alpha\theta_m})e^{\alpha\theta_m} - f(\sigma_0 e^{\alpha\xi})e^{\alpha\xi}}{\left(\int_{\sigma_0 e^{\alpha\xi}}^{\sigma_0 e^{\alpha\theta_m}} f(\tau)\,d\tau\right)^{3/2}}\, d\xi. \tag{39}$$

Since (1) f is monotone, (2) $\sigma_0 > 0$, and (3) $0 < \xi < \theta_m$, it may be concluded that the numerator of the integrand in (39) is positive. Thus, $\partial T/\partial \sigma_0 < 0$.

By monotonicity $f(z) \geq \varepsilon$ for z sufficiently large. Therefore,

$$T(\theta_m, \sigma_0) \leq \frac{1}{\sqrt{\varepsilon\sigma_0}}\int_0^{\theta_m} \frac{d\xi}{(e^{\alpha\theta_m} - e^{\alpha\xi})^{1/2}} \tag{40}$$

for σ_0 sufficiently large. The integral appearing in (40) is finite, so

$$\lim_{\sigma \to \infty} T(\theta_m, \sigma) = 0.$$

Similarly, because $f(0) = 0$, we have that $f(z) \leq M|z|$ for z sufficiently small. Therefore

$$T(\theta_m, \sigma_0) \geq \int_0^{\theta_m} \frac{d\xi}{\left(\int_0^{\sigma_0 e^{\alpha\theta_m}} f(\tau)\,d\tau\right)^{1/2}} \geq \frac{\theta_m}{\sqrt{M\sigma_0 e^{\alpha\theta_m}}}, \tag{41}$$

for σ_0 sufficiently small and positive, and consequently

$$\lim_{\sigma_0 \searrow 0} T(\theta_m, \sigma_0) = \infty.$$

□

Lemma 3. *The function* $\sigma_0(\theta_m)$ *satisfies*

$$\lim_{\theta_m \searrow 0} \sigma_0(\theta_m) = 0, \quad \lim_{\theta_m \to \infty} \sigma_0(\theta_m) = 0. \tag{42}$$

Proof. We proceed by contradiction in the case that $\theta_m \to \infty$. The other case is similar. If the function $\sigma_0(\theta_m)$ does not tend to zero there will be an infinite sequence θ_m^n with the property $\sigma_0(\theta_m^n) > \delta$ for some constant δ, and $\lim_{n\to\infty} \theta_m^n = \infty$. Monotonicity of f implies that on this sequence $\sigma_0(\theta_m^n)$

$$\int_{\sigma_0 e^{\alpha\zeta}}^{\sigma_0 e^{\alpha\theta_m^n}} f(\tau)\,d\tau \geq \int_{\delta e^{\alpha\zeta}}^{\delta e^{\alpha\theta_m^n}} f(\tau)\,d\tau.$$

Because $f(p)$ is strictly positive and monotone, there also exists a positive constant ε such that $f(z) \geq \varepsilon$ for all $z \geq \delta$. Consequently from (36) for each θ_m^n we have the estimate

$$T(\theta_m^n, \sigma_0(\theta_m^n)) \leq \frac{1}{\sqrt{\varepsilon\delta}} \int_0^{\theta_m^n} \frac{d\xi}{(e^{\alpha\theta_m^n} - e^{\alpha\xi})^{1/2}}. \tag{43}$$

But it can be calculated that in the limit $n \to \infty$ and $\theta_m^n \to \infty$ the right-hand-side of (42) approaches zero. This limiting behavior provides a contradiction because, by definition of $\sigma_0(\theta_m^n)$, the time map $T(\theta_m^n, \sigma_0(\theta_m^n))$ is a nonzero constant independent of n. □

In order to exploit the stability-prediction theorems that were described in Section 3, we must construct the distinguished bifurcation diagram. The general theory will be applied in the specific case that arises when integral (10) is identified with functional (21), and σ_0 plays the role of the parameter γ. The distinguished functional is then

$$-I_{\sigma_0} = \int_{-1}^{1} v_x(x)\,dx = v(1) - v(-1) = v(1) \equiv v_0, \tag{44}$$

where the boundary condition $v(-1) = 0$ has been invoked. Accordingly we wish to construct the (σ_0, v_0) bifurcation diagram. Notice from boundary conditions (7) or (8) that either σ_0 or v_0 is prescribed, and the other is to be determined. However we wish to find all solutions as both σ_0 and v_0 are varied. Consequently the *set* of all solutions is the same for both boundary-value problems.

It has already been determined that $\theta_m \in [0, \infty)$ provides a parametrization of the unique branch of solutions, and the qualitative features of the dependence of σ_0 on θ_m have been found. The final result of this section will describe the dependence of v_0 on θ_m. This dependence completes the qualitative information required to determine stability properties purely in terms of the shape of the solution curve in the distinguished bifurcation diagram.

Lemma 4. *The function $v_0(\theta_m)$ is monotone increasing, and satisfies*

$$v_0(0) = 0, \quad \lim_{\theta_m \to \infty} v_0(\theta_m) = \infty. \tag{45}$$

Proof. It has already been shown that $\sigma_0(0) = 0$. Accordingly (28) and constitutive hypotheses (26) imply that when $\theta_m = 0$, $v_x(x) \equiv 0$. Therefore $v_0(0) = v(-1) = 0$.

Next we note that eqns (27) and (28) imply the identity

$$\theta_{xx} + \kappa\sigma_0 v_x = 0. \tag{46}$$

Integration with respect to x, followed by exploitation of boundary conditions and symmetry of the phase-plane, then yields the expression

$$v_0(\theta_m) = \frac{2\theta_x(-1)(\theta_m)}{\kappa\sigma_0(\theta_m)}. \tag{47}$$

Because $\sigma_0 \to 0$ in the limit $\theta_m \to \infty$, and because inspection of the phase-plane reveals that $\theta_x(-1)$ is bounded away from zero in the same limit, it can be concluded that $\lim_{\theta_m \to \infty} v_0(\theta_m) = \infty$.

It only remains to prove that $v_0(\theta_m)$ is a monotone increasing function. However our proof will involve a different parametrization. Let a new variable $\psi(x)$ be defined in terms of $\theta(x)$ and σ_0 through

$$\psi(x) = \theta(x) + \frac{1}{\chi}\log\sigma_0. \tag{48}$$

Then the maximum value of ψ, namely ψ_m, arises at $x = 0$, and is determined by θ_m. We will show that ψ_m induces a new parametrization of the solution curve. To see this notice that the function $\psi(x)$ satisfies the differential equation

$$\psi_{xx} + \kappa e^{\chi\psi} f(e^{\chi\psi}) = 0, \tag{49}$$

[cf. eqn (30)] subject to the boundary conditions

$$\psi(-1) = \psi(1) = \frac{1}{\chi}\log\sigma_0, \tag{50}$$

[cf. boundary conditions (32)]. In contrast to the phase-plane (31) for (θ, θ_x), the phase-plane

$$\frac{1}{2}\frac{\chi}{\kappa}\psi_x^2 + F(e^{\chi\psi}) = A \tag{51}$$

of (ψ, ψ_x) does not depend on σ_0. Rather the dependence on σ_0 has been transferred to the boundary conditions. To see that ψ_m parametrizes the curve of solutions we merely integrate (49) with initial data $\psi(0) = \psi_m$, $\psi_x(0) = 0$ backwards in x from $x = 0$ to $x = -1$. Then the value $\psi(-1)$, symmetry of the phase-plane, and boundary conditions (50) will uniquely determine the appropriate value of σ_0 for a solution.

The remainder of the proof is completed in two steps. First it will be shown that the two parametrizations θ_m and ψ_m are one-to-one, so that θ_m increases monotonically with ψ_m. Then a phase-plane argument will be used to analyse monotonicity properties of $v_0(\psi_m)$, which will imply the required monotonicity properties of $v_0(\theta_m)$.

Suppose that the two parametrizations θ_m and ψ_m are not one-to-one, so that there exist two values $\theta_m \neq \hat{\theta}_m$, with $\psi_m = \hat{\psi}_m$. Then the associated functions $\psi(x)$ and $\hat{\psi}(x)$ both satisfy eqn (51) for some values of the integration constant A and $\hat{A}$. But the maximum values of $\psi(x)$ and $\hat{\psi}(x)$ are equal and are both achieved at $x = 0$, so $\psi_x(0) = \hat{\psi}_x(0) = 0$, and consequently $A = \hat{A}$. Therefore $\psi(x) \equiv \hat{\psi}(x)$. From the definition (48) of ψ we conclude that $\theta(x) = \hat{\theta}(x) + 1/\chi(\log\hat{\sigma}_0 - \log\sigma_0)$. And from the boundary conditions (32) we have that $\theta(1) = \hat{\theta}(1) = 0$, so it may be concluded that $\sigma_0 = \hat{\sigma}_0$. Thus $\theta(x) \equiv \hat{\theta}(x)$, and in particular $\theta_m = \hat{\theta}_m$ which provides a contradiction. Accordingly it may be concluded that θ_m and ψ_m are in a one-to-one relation.

Representation (47) of v_0 can now be rewritten in the form

$$v_0(\psi_m) = \frac{2\theta_x(-1)(\psi_m)}{\kappa\sigma_0(\psi_m)}. \tag{52}$$

Moreover the requirement that the time-map along trajectories in the phase-plane (51) have value 1 can be written as

$$\frac{2\chi}{\kappa}\int_0^{\psi_x(-1)} \frac{d\xi}{e^{\chi\psi} f(e^{\chi\psi})} = 1, \tag{53}$$

where to perform the integration in (53), ψ must be eliminated in favor of the variable ξ

through the first-integral $\frac{1}{2}(\alpha/\kappa)\xi^2+F(e^{\alpha\psi}) = A$. For our purposes the given form of (53), and monotonicity of the constitutive function f suffice to demonstrate that $\psi_x(-1) = \theta_x(-1)$ is a monotonically increasing function of the parameter ψ_m and therefore of the parameter θ_m.

Representation (52) now reveals that $v_0(\psi_m)$ is monotone increasing on any segment for which $\sigma_0(\psi_m)$ is decreasing. That is v_0 is an increasing function of ψ_m, and therefore also of θ_m, on any backward-going segment of the solution branch in the bifurcation diagram (cf. Fig. 4b). In the next section we shall prove that all forward-going segments represent local minima. The general result described in Lemma 1 then implies that $v_0(\theta_m)$ is also an increasing function on forward-going segments. The combination of these two arguments demonstrate that $v_0(\psi_m)$ is monotone increasing on the entire branch, as was required. (This argument has the possibility of being circular because it appears that we here use stability properties to prove monotonicity, and monotonicity to prove stability in Section 5. However the argument detailed in the next section is logically correct. We start from a particular extremal that is shown to be stable by direct calculation. Lemma 1 then applies to prove monotonicity on the first forward-going segment. The argument given here proves monotonicity on the first backward-going segment, which implies, by general arguments, stability of any second forward-going segment, etc.) □

It should be remarked that the parametrization ψ_m introduced in the proof of Lemma 4 could have been adopted throughout the analysis of this section. Some parts of the analysis would thereby have been simplified and others complicated. We have chosen to favor the parametrization by θ_m, in part because of its direct physical interpretation as maximum temperature.

5. THE STABILITY ANALYSIS

In this section the theory of Section 3 is applied to the variational characterization (10) of the steady-state solutions to (5)–(8), and those extremals that actually realize local minima are identified. The implications of this static analysis for stability properties in the dynamic sense of Lyapunov are also discussed.

The first step is to verify the hypotheses upon which the general theory of Section 3 is predicated. In the specific context of the variational principles developed in Section 2 and Section 4, the phase-plane analysis yields much of the required information. In particular θ_m provides a smooth parametrization of a unique solution branch, and differentiation of (33) demonstrates that the parametrization is nonsingular. Furthermore, the Lemmata of Section 4 show that the projection of the solution branch onto the distinguished (σ_0, v_0) bifurcation diagram is a smooth, uncusped curve.

It only remains to verify Legendre's strengthened condition (18), and to ascertain whether $\mu = 0$ can be a multiple eigenvalue of (19). When functional (16) is of the particular form (16), with Σ satisfying constitutive law (14), we have that

$$F_{pp} = \begin{bmatrix} \alpha/\kappa & 0 \\ 0 & e^{-2\alpha\theta}G''(e^{-\alpha\theta}v_x) \end{bmatrix}. \tag{54}$$

Consequently, in light of constitutive hypotheses (25), condition (18) is satisfied only provided that $\alpha > 0$. Jacobi's eigenvalue problem (19) at a solution $(\hat{\theta}, \hat{v}_x)$ is

$$-\frac{d}{dx}\left\{\begin{bmatrix} \alpha/\kappa & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} \phi_x \\ \psi_x \end{bmatrix} - \alpha\begin{bmatrix} 0 & 0 \\ (\sigma_0+\hat{v}_x A) & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \psi \end{bmatrix}\right\}$$
$$-\alpha\begin{bmatrix} 0 & (\sigma_0+\hat{v}_x A) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_x \\ \psi_x \end{bmatrix} + \alpha^2\begin{bmatrix} (\hat{v}_x\sigma_0+\hat{v}_x^2 A) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mu\begin{bmatrix} \phi \\ \psi \end{bmatrix} \tag{55}$$

where the notation $A \equiv e^{-2\alpha\theta}G''(e^{-\alpha\theta}\hat{v}_x)$ has been introduced. The eigenfunctions $(\phi(x), \psi(x))$ must satisfy the linearization of the boundary conditions appearing in (10), namely

$$\phi(-1) = \phi(1) = \psi(-1) = 0. \tag{56}$$

In addition there is the linearization of the natural, stress boundary-condition, namely

$$\psi_x(1) = 0. \tag{57}$$

Lemma 5. *If* $\mu = 0$ *is an eigenvalue of* (55) *subject to boundary conditions* (55) *and* (65), *it is simple.*

Proof. Suppose that (ϕ^0, ψ^0) is a pair of eigenfunctions of (55) with $\mu = 0$ as eigenvalue. Then the second differential equation in (55) can be integrated, and, in light of boundary conditions (56) and (57),

$$e^{-2\alpha\theta}G''(e^{-\alpha\theta}\bar{r}_x)\psi_x^0(x) = \alpha(\sigma_0 + e^{-2\alpha\theta}\bar{r}_x G''(e^{-\alpha\theta}\bar{r}_x))\phi^0(x).$$

Consequently, $\psi_x^0(x)$ can be eliminated from the first equation in (55), and $\phi^0(x)$ can be seen to be an eigenfunction of a second-order, symmetric eigenvalue problem with separated boundary conditions. Such problems have simple eigenvalues. □

The implications of the analysis of Section 3 are now considered. The first step is to determine the number of negative eigenvalues of (55)–(57) at one particular extremal. The simplest candidate is $\theta_m = 0$. Then $\sigma_0 = 0$ and $\bar{r}_x(x) \equiv 0$. It is apparent that for these coefficients (55) has only positive eigenvalues. The stability properties of all other extremals, subject to either stress or velocity boundary conditions, can now be ascertained with no further concrete calculation.

Stress boundary conditions (7) will be considered first, and the bifurcation diagrams depicted in Figs 3 and 4(b) will be referred to throughout. The properties of extremals will be examined as the parameter θ_m is increased, and, according to Lemma 4, the bifurcation branch is traversed upward. Because the extremal at $\theta_m = 0$ has been shown to be a local minimum of (10), and because the phase-plane analysis shows that there is no other solution bifurcating from the primary branches of solutions, the classic principle of exchange of stability applies to provide the information that all of the extremals on the lowest forward-going segment are local minima. That is, eigenvalue problem (55)–(57) at any extremal on the lowest segment has only positive eigenvalues. The classic principle of exchange of stability also implies that there is a negative eigenvalue everywhere on the lowest, backward-going segment, and consequently the associated extremals are not local minima. However for parameter ranges where the bifurcation curve exhibits multiple folds, the classic results can give no information beyond the second fold point.

The analysis of Section 3 provides the same predictions as the classic results up to the second fold point. However it also yields information for the remainder of the solution branch. Because r_0 is monotonically increasing on the backward-going segment (Lemma 4), the upper branch of the first (sub-critical) fold is necessarily connected to the lower branch of the second (super-critical) fold. Accordingly the results of Section 3 apply to state that an eigenvalue passes from the negative half-line to the positive half-line as the second fold is traversed upward. As there was only one negative eigenvalue on the backward-going segment, it may be concluded that on the second forward-going segment there is no negative eigenvalue. Consequently the entire segment represents local minima, and r_0 is monotone increasing. The arguments then repeat at any subsequent folds. In summary, it has been demonstrated that for the case of stress boundary conditions (7), each forward going segment represents local minima. Moreover on each backward going segment, eigenvalue problem (55)–(57) has precisely one negative eigenvalue, and consequently the associated extremals are not minima.

The velocity boundary conditions (8) will now be considered. Then boundary condition (57) is no longer appropriate and must be replaced by

$$\psi(1) = 0. \tag{58}$$

However the eigenvalue problem need not be re-examined. Instead the velocity boundary condition

$$v_0 = v(1) = v(1) - v(-1) = \int_{-1}^{1} v_x(x)\,\mathrm{d}x, \tag{59}$$

can be treated as an additional isoperimetric side-constraint, and (58) can be regarded as the linearization of (59). The example exactly fits that part of the general framework described in Section 3 which concerns differences in stability properties under hard and soft loading. The theory of Maddocks (1987) can be applied, and, for the particular problem at hand, it may be concluded that *all* extremals are constrained local minima. The pertinent observations are as follows. Eigenvalue problem (55)–(57) has at most one negative eigenvalue. When there is no negative eigenvalue, the extremal is a local minimum, and therefore, *a fortiori*, it is a constrained local minimum. A negative eigenvalue occurs only on backward going segments for which σ_0 is decreasing with θ_m. Because v_0 always increases with θ_m it may be concluded that

$$\frac{\partial v_0}{\partial \sigma_0} < 0, \tag{60}$$

which is precisely condition (25). Consequently, for velocity boundary conditions (8), the entire branch of solutions represent local minima.

The property of being a local minimum will now be related to dynamic stability properties. The connection is a standard one, namely Lyapunov's direct method. When constitutive relation (15) is assumed, the functional I defined by (10) is a Lyapunov functional on smooth solutions of dynamical system (5). Differentiation of I with respect to t yields:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_{-1}^{1} \left[\frac{\alpha}{\kappa}\theta_x\theta_{xt} + \{-\alpha\theta_t \mathrm{e}^{-\alpha\theta} v_x + \mathrm{e}^{-\alpha\theta} v_{xt}\} G' - \sigma_0 v_{xt}\right] \mathrm{d}x. \tag{61}$$

Integration by parts on the terms involving xt derivatives, and use of boundary conditions (7) or (8), then provides the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_{-1}^{1} \frac{\alpha}{\kappa}\theta_t(-\theta_{xx} - \kappa \mathrm{e}^{-\alpha\theta} v_x G')\,\mathrm{d}x - \int_{-1}^{1} v_t(\mathrm{e}^{-\alpha\theta} G')_x\,\mathrm{d}x. \tag{62}$$

But θ and v are solutions of the dynamical system (5), so (62) is equivalent to

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\int_{-1}^{1} \left[\frac{\alpha}{\kappa}\theta_t^2 + v_t^2\right] \mathrm{d}x. \tag{63}$$

It is apparent from (63) that provided $\alpha > 0$ the derivative of I is negative along non-steady-state trajectories of (5). Consequently (10) is a candidate for a Lyapunov functional.

The above calculations suffice to prove nonlinear dynamic stability in the sense of Lyapunov for extremals that are local minima, *provided that a global existence result is assumed for the time-dependent initial-boundary-value problem.* While the static or energy criterion for stability is widely accepted, it is important to realize that from a mathematical point of view our analysis is incomplete until a global existence theorem has been obtained for a class of initial data close to the extremal whose stability is in question. Questions of existence cannot be completely disregarded on physical grounds. For example, with nonlinear constitutive laws, the possibility of the formation of shocks or other singularities must be precluded. Moreover the existence result must be in a sufficiently regular class of functions that the formal manipulations of the Lyapunov function performed above can be justified. The derivation of such a result is likely to be difficult, and highly technical, and it will not be pursued here. The most closely-related existence results known to us are given in Dafermos and Hsiao (1982) and Tzavaras (1987). In the former, the existence of global smooth solutions to the equations of one-dimensional thermo-visco-elasticity is proven in the case of stress-free and insulated boundary conditions and under the additional assumption that the stress depends linearly on the strain-rate. In the latter paper a local existence theorem is proven for thermally non-conducting materials (i.e. $\lambda = 0$ in (1)). Although these papers exploit restrictive assumptions not satisfied by our model, they indicate the general techniques and approaches that would likely have to be adopted in order to obtain any analogous theorem for the system (5)–(8). The recent work of Charalambakis and Murat (1989) extends the same line of analysis, and obtains existence of global weak solutions in the adiabatic special case of eqns (1) that arises when $\lambda = 0$.

6. SUMMARY AND CONCLUSIONS

In this paper we have analysed a model of thermo-plastic materials in which the stress $\sigma(\theta, v_x)$ is determined by a constitutive law of the form

$$\sigma(\theta, v_x) \equiv e^{-\alpha\theta} g(e^{-\alpha\theta} v_x), \tag{64}$$

where $g(p)$ is any smooth function satisfying constitutive hypotheses (26). We believe hypotheses (25) to be both mild and physically reasonable for deformations at high strain rates for which elastic effects may reasonably be neglected. In Part II of this paper we shall extend our analysis to include constitutive relations that incorporate a yield stress that models residual elastic effects. To our knowledge there is little experimental evidence either to confirm or refute the particular temperature dependence of σ implied by (64). Wright (1987) discusses such issues and describes alternative models. While the constitutive relation (64) appears to be somewhat special we remark that it contains an arbitrary function $g(\cdot)$ which need only satisfy certain smoothness and growth hypotheses. In this sense the class of constitutive relations falling within the scope of our stability analysis is comparatively large. Prior stability analyses have assumed some specific form of constitutive relation containing only a few arbitrary constants. The form of constitutive relation (64) certainly captures the analyses of most prior authors.

For our purposes the key feature of constitutive relations of the type (64) is that they are intimately related to the existence of a potential. A useful analogy can be drawn with elasticity, where certain elastic constitutive relations can be associated with a strain energy, in which case the material is sometimes described as hyper-elastic. Thus materials satisfying (66) might be described as hyper-thermo-plastic. (However we personally do not intend to promulgate such jargon.) For such materials the steady-state solutions of the dynamical system (5)–(8) can be characterized as extremals of the functional

$$I(\theta, v_x, \sigma_0) = \int_{-h}^{h} \left[\frac{1}{2} \frac{\alpha}{\kappa} \theta_x^2 + G(e^{-\alpha\theta} v_x) - \sigma_0 v_x \right] dx, \tag{65}$$

where G denotes the anti-derivative of the constitutive function g.

Steady-state solutions of (5)–(8) satisfy a two-point boundary-value problem for a system of two, coupled, second-order, ordinary differential equations. In Section 4 it was shown that solutions of this system are completely described by the single phase-plane (31) (cf. Fig. 2). Analysis of this phase-plane demonstrates that the two-point boundary-value problem possesses a unique one-dimensional family of solutions that can be parametrized by the maximum temperature θ_m of the solution. In the case that $G(p)$ is quadratic, $g(p)$ is linear, and the stress σ has a linear dependence on strain-rate v_x. The bifurcation diagram can then be found in closed form [cf. (34)], and there is only one fold in the branch of

(a)

(b)

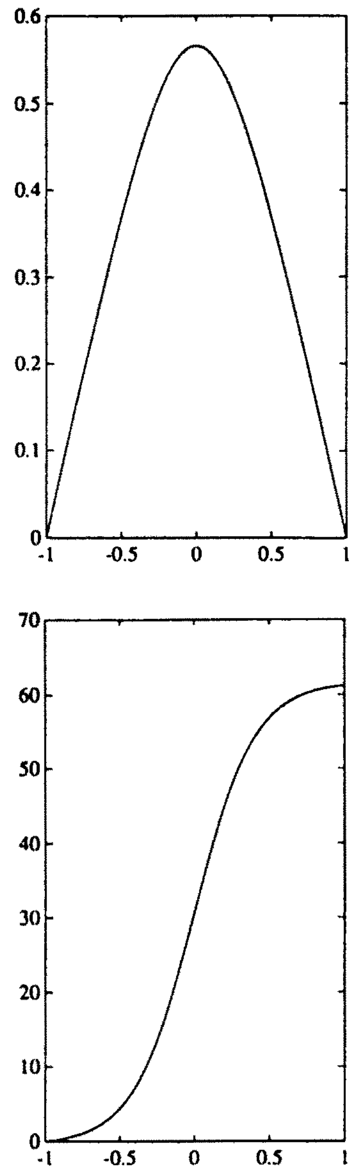

Fig. 5. The temperature and velocity profiles of the solution corresponding to the point $(\sigma_0, v_0) = (1.263, 61.236)$ in the bifurcation diagram depicted in Fig. 3. The abscissa is x. In part (a) the ordinate is temperature θ, and in part (b) the ordinate is velocity v.

solutions (cf. Fig. 3). For more general constitutive laws the exact form of the bifurcation diagram must be calculated numerically. For example for the constitutive relation depicted in Fig. 4(a), the bifurcation diagram is shown in Fig. 4(b) (the different branches correspond to different values of the constant on the right-hand-side of (37), which corresponds physically to different combinations of the material parameters). These solution branches were found by numerically integrating the differential equation (48) from initial conditions $(\psi(0), \psi_x(0)) = (\psi_m, 0)$, and the values of σ_0 and v_0 are then determined from $(\psi(1), \psi_x(1))$ by using eqns (50) and (47). It is apparent that the solution branch may or may not have multiple folds. It is interesting that the existence and the number of multiple folds seem to be related to the relative sizes of the regions of convexity and concavity of f.

Figure 5 shows the temperature and velocity profiles of a steady-state solution corresponding to the constitutive law described in the caption of Fig. 1. These data are given here merely to confirm the general properties of solutions that were predicted. The temperature is maximal in the center of the band, as is the velocity gradient v_x. We do not claim that these particular solutions provide compelling evidence for shear-bands, but they do possess the correct qualitative features. Moreover the velocity profile steepens quickly as the branch of solutions is traversed outward. Nevertheless a detailed numerical study of the solution profiles is a work unto itself and we do not perform it here.

The stability properties of all solutions have been determined, whether the constitutive law gives rise to a single fold or multiple folds. In the case of stress boundary conditions (7), the solution branch that passes through the origin is stable up to the first fold point. Stability is lost for the backward-going segment that lies between the first and second folds, and is regained at the third fold point. Stability is then successively lost and regained at any further fold points. Stability is here used in the sense that the steady-state solution in question realizes a local minimum of the potential (65). According to the theory of Maddocks (1987) that is summarized in Section 3, such stability results can be obtained immediately once the qualitative properties of the (σ_0, v_0) bifurcation diagram have been ascertained either analytically or numerically.

In the case of velocity boundary conditions (8), all solutions are stable. The differences in stability properties between stress and velocity boundary conditions are explained by the fact that the appropriate variational principle for velocity boundary conditions involves a side constraint. In order to be stable for velocity boundary conditions, an extremal need only realize a conditional local minimum, which is a less stringent requirement than that which arises for stress boundary conditions. The theory of Maddocks (1987) again allows a classification of conditional local minima to be obtained directly from the distinguished (σ_0, v_0) bifurcation diagram. We remark that the solution shown in Fig. 5 is actually unstable under stress boundary conditions (7) and stable under velocity boundary conditions (8).

To our knowledge all of the above results are new. A direct analysis of linear stability would involve linearization of the system (5) and separation of the time variable. For general constitutive relations the eigenvalue problem so obtained would give rise to a non-self-adjoint system of ordinary differential equations with nonconstant coefficients. The difficulties in estimation of eigenvalues of such problems should not be underestimated. Prior analyses have only been carried out for very particular constitutive laws, and with certain other simplifying assumptions. Two such assumptions are to either freeze the coefficients and so obtain a constant coefficient eigenvalue problem, or to neglect acceleration terms in which case the order of the differential system is decreased. The relationship between the eigenvalues of either of these simplified problems, and those of the full eigenvalue problem is unclear. Stability analyses of the problem with velocity boundary conditions are particularly rare. The only other such analysis known to us was carried out by Molinari and Leroy (1990) with a separation of variables argument. They assume a simpler class of constitutive relations than that allowed here, and make the quasistatic assumption in which acceleration terms are neglected. A byproduct of the approach outlined in this paper is the unification of the stability analyses of the problems with stress and velocity boundary conditions.

Because the time-derivative terms enter (5) in a particularly simple fashion, it is apparent that our steady-state, variational characterization of stability properties coincides with predictions that would arise from the eigenvalue problem that is obtained from linearization of the dynamical system followed by separation of the time variable. However it is possible to do better than that. In Section 5 it was shown that the potential (65) is a Lyapunov function for the full nonlinear dynamics (5)–(8). The characterization of stable and unstable steady-state solutions purely in terms of the type of the associated critical point of the potential is thereby justified, provided only that an appropriate time-dependent existence theorem is proven for the governing system of partial differential equations. However such an existence result is not available at this point, although the recent theorem of Charalambakis and Murat (1989) points to the right tools and techniques needed in producing a theorem on the existence of global weak solutions for our problem. Accordingly, when the constitutive relation for the stress σ is of the form (15), we believe that our analysis resolves questions of stability of the steady-state solutions of (5)–(8) to the full extent currently possible. Nevertheless questions of time-dependent existence remain. The nonvariational case which arises for constitutive relations more general than (15) is also an open problem, although some partial results are obtained in Malek-Madani *et al.* (1991).

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